

Statistical predictability in two-dimensional turbulence

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The predictability problem in the inverse energy cascade of two-dimensional turbulence is addressed by means of high resolution direct numerical simulations. The analysis is done in terms of the finite size Lyapunov exponent (FSLE) which is a measure of the growth rate at fixed error level. The numerical results are compared with classical closure predictions and good quantitative agreement is found. Finally, it is shown that the inertial range predictability properties are not affected by the presence of noise induced by the small, not resolved, scales.

I. INTRODUCTION

The knowledge of predictability properties of two-dimensional turbulence is fundamental for estimating the predictability of atmospheric flow (Leith 1971, Leith and Kraichnan 1972). The characteristic property of two-dimensional flow is enstrophy conservation which force the energy to flow toward large scales. In this inverse cascade regime, dimensional arguments predict a “5/3” Kolmogorov energy spectrum which is indeed observed in the large-scale atmospheric spectrum (Nastrom, Gage and Jasperson 1984).

From a theoretical point of view, predictability in fully developed turbulence has been investigated as a prototypical model with many characteristic scales and time. The first attempts to the study of predictability in turbulence dates back to the pioneering work of Lorenz (Lorenz 1969) and to Kraichnan and Leith papers (Leith 1971, Leith and Kraichnan 1972). On the basis of closure approximations, they were able to obtain quantitative predictions on the evolution of the error in different turbulent situations, both in two and three dimensions. Their fundamental papers become the backbone for more recent approaches (Lesieur 1997). Because predictability experiments in fully developed turbulence are numerically very expensive, to our knowledge there are still no attempt to compare closure results with direct numerical simulations.

In this paper we address the predictability problem for inverse energy cascade of two-dimensional turbulence by means of high resolution direct numerical simulations. It has been recently shown that inverse cascade is not affected by intermittency corrections and velocity statistics is quasi-Gaussian (Paret and Tabeling 1997, Boffetta, Celani and Vergassola 2000). This makes the problem simpler than the three-dimensional case and we expect that closure-based predictions are essentially correct.

At variance with direct cascade, the inverse energy cascade cannot be observed in decaying turbulence. In order to sustain the cascade, a continuous input of energy by random forcing at small scales is necessary. Because we are interested in the intrinsic predictability of the model, we will study the problem assuming known the realization of the forcing. In realistic applications, the forcing represents the small scale dynamics not resolved by the two-dimensional model (i.e. convective motion in the atmosphere). In this case one should take into account also the uncertainty introduced by the random forcing. Because of the hierarchical structure of the characteristic times, we will see that the inertial range predictability properties are not affected by the uncertainty introduced by the forcing.

This remainder of the paper is organized as follows: Section II is devoted to a brief summary of the classical closure results. In Section III and we present the numerical results and their comparison with predictions. Section IV is devoted to conclusions.

II. STATISTICAL TURBULENCE PREDICTABILITY

Given two realizations of the velocity field $\mathbf{u}^{(1)}(\mathbf{x}, t)$ and $\mathbf{u}^{(2)}(\mathbf{x}, t)$, a suitable measure for the predictability is the error field

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$$\delta \mathbf{u}(\mathbf{x}, t) = \frac{1}{\sqrt{2}} \left(\mathbf{u}^{(2)}(\mathbf{x}, t) - \mathbf{u}^{(1)}(\mathbf{x}, t) \right) \quad (1)$$

from which we define the error energy and the error energy spectrum as (Leith and Kraichnan 1972, Lesieur 1997)

$$E_{\Delta}(t) = \frac{1}{2} \int |\delta \mathbf{u}(\mathbf{x}, t)|^2 d^2 x = \int_0^{\infty} E_{\Delta}(k, t) dk. \quad (2)$$

Normalization in (1) ensures that $E_{\Delta}(k, t) \rightarrow E(k)$ for uncorrelated fields (i.e. for $t \rightarrow \infty$).

The two realizations of the turbulent flow are stationary solutions of two-dimensional Navier Stokes equation in which the inverse cascade is sustained by energy injection at small-scale wavenumber k_f with a Gaussian random forcing. As it is customary in predictability experiments, for most of the results presented here the two realizations of the forcing will be the same. In this case the growth of the error will be uniquely due to the deterministic chaotic dynamics. At the end of Section III we will discuss the case in which two different realizations of the forcing are implemented.

Assuming that the initial error can be considered infinitesimal, the magnitude of the difference field starts growing exponentially and $E_{\Delta}(t) \simeq E_{\Delta}(0) \exp(L(2)t)$ where $L(2)$ is the generalized Lyapunov exponent (in non-intermittent systems, $L(2) = 2\lambda$, see (Bohr, Jensen, Paladin and Vulpiani 1998)). The error growth in this stage is confined at the faster scales in the inertial range, corresponding in our situation to the scales close to the forcing wavenumber k_f , while at larger scales the two fields remain correlated (see Figure 1). At larger times, when $E_{\Delta}(k_f, t)$ becomes comparable with $E(k_f)$, the exponential growth terminates, because the two fields are completely decorrelated at small scales. The error growth continues at larger scales in the inertial range, where the two fields are still correlated, and the time evolution of the error follows an algebraic law.

The error transfer towards large scale can be quantified by the characteristic wavenumber $k_E(t)$ at which the error spectrum is a given fraction of the reference spectrum. Leith and Kraichnan (Leith and Kraichnan 1972), defining k_E such that the relative error spectrum $r(k, t) = E_{\Delta}(k, t)/E(k)$ is $r(k_E) = 0.5$, found a self-similar error spectrum $r(k/k_E)$ in which the time dependence is only through $k_E(t)$.

The dimensional prediction, based on the assumption that the time it takes for the error to induce a complete uncertainty at wavenumber k is proportional to the characteristic time at that scale gives, within the Kolmogorov scaling

$$\frac{dk_E}{dt} = B\varepsilon^{1/3} k_E^{-5/3} \quad (3)$$

where B is an adimensional constant. By integration one gets

$$k_E(t)^{-2/3} = k_0^{-2/3} + B\varepsilon^{1/3}(t - t_0) \quad (4)$$

If the inertial range is sufficiently wide, we can assume that $k_0 \ll k_E$ and thus $k_E(t) \sim \varepsilon^{-1/2} t^{-3/2}$ (Lorenz 1969). At wavenumbers smaller than k_E the error is still small in comparison with the typical energy, while at larger wavenumbers the two fields are completely decorrelated. Thus one can assume that in (2) one has

$$E_{\Delta}(k) = \begin{cases} E(k) & \text{for } k > k_E(t) \\ 0 & \text{for } k < k_E(t) \end{cases} \quad (5)$$

Using the Kolmogorov spectrum for $E(k)$ one ends with the prediction

$$E_{\Delta}(t) = G\varepsilon t \quad (6)$$

The numerical constant G in (6) can be obtained only by repeating the argument more formally within a closure framework. In the case of two-dimensional inverse cascade the Test Field model predicts $G \simeq 4.19$ (Leith and Kraichnan 1972).

Under the hypothesis of an initial error localized on infinitesimal scale in a wide inertial range (i.e. $k(0) \rightarrow \infty$), reverting equation (4) is possible to relate the predictability time on a certain scale with the characteristic time taken from the energy spectrum:

$$T_p(k) = \frac{1}{B} \varepsilon^{-1/3} k^{-2/3} = \frac{C^{1/2}}{B} [E(k)k^3]^{-1/2} \equiv \frac{C^{1/2}}{B} \tau(k) \quad (7)$$

where C is Kolmogorov constant ($C \simeq 6$ in two dimension) and $\tau(k)$ is the spectrum-base characteristic time. This result has a practical utility, because in atmospheric turbulence is obviously impossible to perform a “predictability experiment” and equation (7) give an estimate for predictability time simply measuring the energy spectrum.

An alternative approach to predictability problem is the recently introduced Finite Size Lyapunov Exponent analysis (Aurell et al. 1996). FSLE is a generalization of the Lyapunov exponent to finite size errors. Within this approach one computes the error doubling time $T_r(\delta)$ which takes for an error of size δ (with a given norm) to grow of a factor r (typically $r = 2$) From the average doubling time one defines the FSLE as

$$\lambda(\delta) = \frac{1}{\langle T_r(\delta) \rangle} \ln r \quad (8)$$

It is easy to show that definition (8) reduces to the standard Lyapunov exponent λ in the limit of infinitesimal errors $\delta \rightarrow 0$ (Aurell et al. 1996). As the Lyapunov exponent can be seen like the inverse of the fastest characteristic time of a dynamical system, the FSLE can be considered the inverse of the characteristic time at a certain scale. For finite errors, the FSLE measures the effective error growth rate at error size δ . Let us remark that taking averages at fixed time, as in (6) is not the same of averaging at fixed error size, as in (8). This is particularly true in the case of intermittent systems, in which strong fluctuations of the error in different realizations can hide scaling laws in time. From a numerical point of view, the computation of $\lambda(\delta)$ is not more expensive than the computation of the Lyapunov exponent with a standard algorithm.

In turbulence predictability, a natural measure of the error is $\delta = \sqrt{2E_\Delta}$. The assumption of locality, i.e. that the FSLE is proportional to the inverse of the characteristic time at the scale k such that the typical velocity is $u(k) \sim \delta$, combined with Kolmogorov scaling for the velocities produces the scaling law for the FSLE

$$\lambda(\delta) = A\epsilon\delta^{-2} \quad (9)$$

The constant A relates the energy flux in the cascade to the rate of error growth. Its value is not determined by dimensional arguments, but is easy to show that in absence of intermittency and for $r \simeq 1$ it can be related to the constant G in (6) by $A = \ln(r^2)/(r^2 - 1)G$. In the limit $r \rightarrow 1$ one gets $A \rightarrow G$.

Because of the appearance of the energy dissipation at the first power in (9), we expect that this scaling law is universal, i.e. not affected by possible intermittency in the velocity statistics (Aurell et al. 1996). The scaling law (9) is valid within the inertial range $u(k_f) < \delta < U$ where $u(k_f)$ represents the typical velocity fluctuation at forcing wavenumber and $U \simeq \sqrt{2E}$ is the large scale velocity. At large errors $\delta \simeq U$, we expect error saturation, $E_\Delta \rightarrow E$ and thus $\lambda(\delta) \rightarrow 0$.

III. NUMERICAL SIMULATIONS AND ANALYSIS

We have performed extensive direct numerical simulation of the two-dimensional Navier-Stokes equation written for the scalar vorticity $\omega(\mathbf{x}, t) = \nabla \times \mathbf{u}(\mathbf{x}, t) = -\Delta\psi(\mathbf{x}, t)$ as

$$\partial_t \omega + J(\omega, \psi) = \nu \Delta \omega - \alpha \omega + f \quad (10)$$

where J represents the Jacobian with the stream function ψ . The friction term in (10) removes energy at large scales: it is necessary in order to avoid Bose-Einstein condensation on the gravest mode and to obtain a stationary state (Smith and Yakhot 1993). Physically this term represents the effect of bottom friction. The random forcing f is δ -correlated in time and injects energy at wavenumber k_f only.

Numerical integration of (10) is performed by a standard pseudo-spectral code fully dealiased with second-order Runge-Kutta time stepping on a doubly periodic square domain with resolution $N = 1024$. As it is customary in numerical simulations, we use hyperviscous dissipation in order to extend the inertial range.

Stationary turbulent flow is obtained after a very long simulation starting from a zero initial vorticity field. At stationarity one observes a wide inertial range with a well developed Kolmogorov energy spectrum $E(k) = C\epsilon^{2/3}k^{-5/3}$ (see Figure 1). Structure functions in physical space are found in agreement with the self-similar Kolmogorov theory and no intermittency is detected (Boffetta, Celani and Vergassola 2000).

The perturbed field is obtained from a configuration of the velocity field $\mathbf{u}^{(1)}(\mathbf{r}, 0)$ in the stationary state,

$$\mathbf{u}^{(2)}(\mathbf{r}, 0) = \mathbf{u}^{(1)}(\mathbf{r}, 0) + \sqrt{2}\delta\mathbf{u}(\mathbf{r}, 0) \quad (11)$$

in which the initial error $\delta\mathbf{u}(\mathbf{r}, 0)$ is very small ($E_\Delta(t=0) \simeq 10^{-5}E$), and is localized on small scales. The error energy spectrum (1) is thus initially localized at wavenumbers greater than the forcing wavenumber. In the inertial range the two fields are completely correlated at $t = 0$. Let us remark that the precise form of the initial error spectrum is not important provided that it can be considered infinitesimal, at it will immediately evolve toward the direction of the first Lyapunov eigenvector.

The two configuration are integrated in time according to (10) and the evolution of the error $\delta \mathbf{u}(\mathbf{r}, t)$ is computed. In this section we present the results obtained using the same realization of random forcing in both the realizations. In this way the growth of the error is only induced by the turbulent dynamic. At the end of this section we will compare this results with the ones obtained with two independent realizations of the forcing.

In Figure 2 we plot the time evolution of the error energy $\langle E_\Delta(t) \rangle$ obtained from direct numerical simulations averaged over 20 realizations. The exponential regime is clearly visible at small times, showing that for infinitesimal error turbulence behaves exactly as a standard chaotic system. At the very beginning is possible to observe an initial recorelation of the two fields, i.e. a decreasing of $E_\Delta(t)$: it takes a small, but finite time for the initial perturbation to align in the direction of the leading Lyapunov exponent. During this time the forcing and dissipation recorelate the two fields.

Figure 3 shows the computation of the FSLE from our simulations. For very small errors, $\delta < u(k_f)$ corresponding to an error spectrum $E_\Delta(k_f, t) \ll E(k_f)$, we observe the convergence of $\lambda(\delta)$ to the leading Lyapunov exponent $\lambda \simeq 1.07$. Its value is essentially the inverse of the smallest characteristic time in the system and represents the growth rate of the most unstable features. At larger errors $\delta > 10^{-2}$, we clearly see the transition to the inertial range scaling (9). At further larger errors $\delta \simeq U \simeq 0.1$, $\lambda(\delta)$ falls down to zero in correspondence of error saturation.

In order to emphasize scaling (9), in Figure 3 we also show the compensation of $\lambda(\delta)$ with $\varepsilon \delta^{-2}$. Prediction (9) is verified with very high accuracy which allows to determine the value of $A = 3.9 \pm 0.1$. With the present value of $r \simeq 1.057$, this corresponds to a value $G = 4.1 \pm 0.1$. The physical picture we obtain is that the creation of uncorrelated energy in the inertial range due to chaotic dynamics is about 4 times faster than the energy transfer rate. Our numerical result is in remarkable agreement with the old prediction obtained within the Test Field Model closure which gives $G = 4.19$ (Leith and Kraichnan 1972).

The physical meaning of G is the ratio of the rate of uncorrelated energy production to the rate of energy injected by the forcing and transfered to large scales ε . The fact that $G > 1$ shows that the uncorrelated energy is not simply “transported” with the energy through the cascade, but there is an effective production of error at each scale, due to the chaotic dynamic. The constant rate of error-energy growth is not an immediate consequence of the constant energy flux in the inverse cascade, but is the effect of the dimensional hypothesis that the times for energy transfer and error growing at a fixed scale should follow the same scaling law. Moreover $G > 1$ suggests that (6) should be unchanged when one considers two independent realizations of the forcing, as we will see below.

The time evolution of the error energy spectrum (Figure 1) shows a sharp front of error that propagates from small to large scales justifying *a posteriori* the assumption (5). From the error spectra in Figure 1 one can compute the characteristic wavenumber $k_E(t)$ defined as $r(k_E) = 0.5$. On scales smaller than $k_E(t)^{-1}$ the two fields are already decorrelated, while on larger scales the error can still be considered small, so we can define implicitly the predictability time $T_p(k)$ at certain wavenumber k from $k_E(T_p) = k$.

Figure 4 shows the evolution of $k_E(t)^{-2/3}$ compared with the best fit based on (4). The result is rather noisy because of the discrete character of the wavenumbers, nevertheless it is possible to estimate the value of the constant in (4) $B = 0.43 \pm 0.02$. In equation (7) B relates the predictability time to the characteristic time based on the spectrum. From our simulation we have $T_p(k) \simeq 5.7\tau(k)$.

In order to check the prediction of the self-similar error growth we rescale all the error spectra with the energy spectrum, obtaining the relative error spectra $r(k, t) = E_\Delta(k, t)/E(k)$. Rescaling the wavenumbers with the characteristic wavenumber $k_E(t)$ the error spectra collapse in a similarity error spectrum $r(k/k_E)$ (Figure 5). This means that in order to characterize statistically the error growth one needs to know only the evolution of $k_E(t)$. Closure computations also predict that the shape of the error spectrum in the limit $k \rightarrow 0$ is $E_\Delta(k, t) \sim k^{-3}$ that for the similarity error spectrum gives $r(x) \sim x^{14/3}$ which is indeed observed in our simulations.

A. Different forcing

All the results presented above are obtained forcing identically the two realizations of the turbulent field. In principle this is not consistent with a realistic application like modeling large scale atmospheric flow because in that case the forcing is due to the motion on small scales that are non resolved. In order to simulate this continuous injection of error it is more correct to use two independent realizations of the random forcing. To check in which way this can affect the results discussed above, we have repeated the same numerical experiment with two independent forcing.

The main difference is that in this case the initial exponential growth of the infinitesimal error is non present. When the scales smaller than the forcing scale are non completely decorrelated the main source of error is due to the different forcing, not to the chaotic dynamic. If there were no dynamical effects we could expect to observe a “diffusive” behavior and the error energy should grow linearly as in (6) but because the source of the error is the energy input one should have $G \simeq 1$. This regime is indeed observed (Figure 3) but is worthwhile to remember that

it is physically of little interest: we are using different forcing to mimic the unresolved small scale motion and so it is natural to assume that the two fields are completely decorrelated at those scales.

When this condition is satisfied, and the error is localized in the inertial range, it is not possible to distinguish the case with a different forcing from the one with one forcing. From the analysis of the FSLE (Figure 3) it is evident that in this range the scaling law (9) is verified with the same value of the constant A . All the inertial range results about the self similar growth of error are recovered. This demonstrate that the continuous injection of error due to an unresolved small-scale motion do not produce effects when the two turbulent fields are already decorrelated on such scales. This is a consequence of locality: the error growth in the inertial range is completely determined by the non linear dynamic, and it is not sensible on what happens on smaller scale.

IV. CONCLUSION

We have studied the predictability problem in the inverse cascade of two-dimensional turbulence. Using the Finite Size Lyapunov Exponent we have shown that after an initial exponential growth, a linear growth of the decorrelated energy sets in. This decorrelated energy flows to large scales in self similar way through an inverse cascade with a constant flux that is about four times faster than the energy flux.

This behavior do not change also adding other possible perturbation on small scale, thus the results obtained in a numerical experiment with initial condition uncertainty can be applied to a wider class of problems. As an example the predictability time of mesoscale in the atmosphere can be estimated easily with a simple measure of the energy spectrum.

All the results presented are in strong agreement with the prediction of closure. At least from the point of view of predictability, two-dimensional turbulence thus seems to be very well captured by low-order closure scheme. As a consequence we can exclude, on the basis of our numerical findings, the existence of intermittency effects in the inverse cascade of error. This is a result which is probably of more general interest than the specific problem discussed in this article.

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FIG. 1. Stationary energy spectrum $E(k)$ (thick line) and error spectrum $E_\Delta(k, t)$ at time $t = 4.6, 5.5, 7.1, 10.0, 15.6$. $k_f = 320$ is the forcing wavenumber. In the inset we plot the compensated spectrum $\epsilon^{-2/3} k^{5/3} E(k)$.

FIG. 2. Error energy $\langle E_\Delta(t) \rangle$ growth averaged over 18 runs. Dashed line represents closure prediction (6), dotted line is the saturation value E . The initial exponential growth is emphasized by the lin-log plot in the inset where the initial decreasing of the error is also observable.

FIG. 3. Finite size Lyapunov exponent $\lambda(\delta)$ as a function of velocity uncertainty δ . In the simulation with identical forcing for the two fields (+) the asymptotic value for $\delta \rightarrow 0$ gives the leading Lyapunov exponent of the turbulent flow. In the case with different forcing (\times) the infinitesimal regime is unphysical. In the inertial range, $\delta \geq 10^{-2}$, the behavior of $\lambda(\delta)$ is identical for both case. Dashed line represent the prediction (9). In the inset we show the compensated plot $\lambda(\delta)\delta^2/\varepsilon$. The line represent the fit to the constant $A \simeq 3.9$.

FIG. 4. Time evolution of the characteristic wavenumber $k_E(t)^{-2/3}$ compared with the dimensional prediction $k_E(t)^{-2/3} = k_0^{-2/3} + B\varepsilon^{1/3}(t - t_0)$ where $t_0 = 6.0$ is the time that the error takes to reach the inertial range at $k_0 = 133.0$

FIG. 5. Similarity error spectrum $r(k/k_E) = E_\Delta(k/k_E, t)/E(k/k_E)$. All the error spectra rescaled collapse together within the error bands. The log-log inset shows the asymptotic behavior $r(x) \sim x^{14/3}$ for $x \rightarrow 0$









